Bosonic and fermionic eigenstates for generalized Sutherland models

2000 J. Phys. A: Math. Gen. 33 3795

(http://iopscience.iop.org/0305-4470/33/20/306)

J. Phys. A: Math. Gen. **33** (2000) 3795–3807. Printed in the UK PII: S0305-4470(00)12681-2

Bosonic and fermionic eigenstates for generalized Sutherland models

Akinori Nishino and Miki Wadati

Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7–3–1, Bunkyo-ku, Tokyo 113–0033, Japan

E-mail: nishino@monet.phys.s.u-tokyo.ac.jp

Received 17 March 2000

Abstract. We construct bosonic and fermionic eigenstates for the generalized Sutherland models associated with arbitrary reduced root systems respectively, through W-symmetrization and W-anti-symmetrization of Heckman–Opdam's nonsymmetric Jacobi polynomials. Square norms of the nonsymmetric Heckman–Opdam polynomials are evaluated from their Rodrigues formulae. The W-symmetrization and W-anti-symmetrization of the nonsymmetric polynomials enable us to evaluate square norms of bosonic and fermionic eigenstates for the generalized Sutherland models.

1. Introduction

In 1971, Sutherland introduced a quantum many-body system on a unit circle ($0 \leqslant \theta_j < 2\pi$) [29, 30],

$$H := -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial\theta_j^2} + \sum_{1\leqslant j<k\leqslant N}\frac{g}{\sin^2(\theta_j-\theta_k)} \tag{1.1}$$

which is now called the Sutherland model. The model has the same number of independent and mutually commutative conserved operators as its degrees of freedom, and therefore is a quantum integrable system. The conserved operators have joint eigenvectors which can be expressed by products of the Jastrow-type wavefunction and the Jack polynomials. The Jack polynomials not only enable us to calculate the exact correlation functions of the model [8, 31] but also provide powerful tools in the theory of condensed matter physics [11].

Quantum mechanical systems which describe many particles with inverse-square-type long-range interactions in a one-dimensional space are, in general, called the Calogero–Sutherland (CS) models [3, 24, 29, 30]. A systematic construction of the commutative conserved operators of the CS models are provided from the Dunkl–Cherednik operator formulations [4, 7, 26]. Generalizations of the formulations are applied to a wide class of the CS models and clarify the relationships with other integrable models [9, 12–14]. Among them, the CS models with trigonometric interactions, for example, the Sutherland model (1.1) are studied in the context of the double affine Hecke algebras. The double affine Hecke algebras were introduced by Cherednik to reconstruct the theories of the Macdonald polynomials [5, 6, 18, 19]. In the differential setting [26], they also give an unified treatment of the conserved operators and the orthogonal polynomials appearing in their eigenstates. Lapointe and Vinet introduced the raising operators which create the bosonic eigenstates of the Sutherland model [15, 16].

As for mathematics, they presented the Rodrigues formula for the Jack polynomial and proved integrality of its coefficients. But its extension to the Sutherland models associated with other root systems has not been established.

In previous papers [21, 23], we presented an algebraic construction of the nonsymmetric Macdonald polynomials and evaluated square norms of the Macdonald polynomials through symmetrization of scalar products of the nonsymmetric Macdonald polynomials. In this paper, we consider bosonic and fermionic eigenstates for the generalized Sutherland models associated with arbitrary reduced root systems through W-symmetrization and W-anti-symmetrization of the nonsymmetric Heckman–Opdam polynomials. This paper is organized as follows: in section 2, we briefly describe the affine root systems and the degenerate double affine Hecke algebras following Opdam and Cherednik. The commutative Dunkl–Cherednik operators are introduced. In section 3, we provide the Rodrigues formulae for the nonsymmetric Heckman–Opdam polynomials and evaluate their square norms. By use of W-symmetrization and W-anti-symmetrization method, we algebraically construct the bosonic and fermionic eigenstates for the generalized Sutherland models and evaluate their square norms in section 4. The final section is devoted to concluding remarks.

2. Degenerate double affine Hecke algebras

2.1. Extended affine Weyl groups

We start with the definition of the extended affine Weyl group which acts on the affine coroot system [5, 10]. Let V be an N-dimensional real vector space with a positive definite symmetric bilinear form $\langle\cdot,\cdot\rangle$. Let $R \subset V$ be an irreducible reduced root system which corresponds to the simple Lie algebra of type A, B, C, D, E, F and G. We take a root basis $\Pi = \{\alpha_i | i \in I\}$ of R where $I = \{1, 2, \ldots, N\}$ a set of indices. A decomposition of R is fixed by the following disjoint union: $R = R_+ \cup R_-$, where R_+ is the set of positive roots relative to Π and $R_- = -R_+$. We denote by $R^\vee (\subset V)$ the coroot system which has the elements $\alpha^\vee := 2\alpha/\langle\alpha, \alpha\rangle$, corresponding to the roots $\alpha \in R$. Let $\Pi^\vee = \{\alpha^\vee | \alpha \in \Pi\}$ be a root basis of $R^\vee$. We define the fundamental weights Λ_i and coweights $\Lambda_i^\vee$ such that $\langle\alpha_i^\vee, \Lambda_j\rangle = \delta_{ij}$ and $\langle\Lambda_i^\vee, \alpha_j\rangle = \delta_{ij}$, respectively. We use the standard notations Q, $Q^\vee$, P and $P^\vee$ for the root lattice, the coroot lattice, the weight lattice and the coweight lattice respectively,

$$
\begin{aligned}
Q &:= \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \\
Q^\vee &:= \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee \subset P^\vee := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i^\vee
\end{aligned}
\tag{2.1}
$$

and Q_+, P_+, $Q_+^\vee$ and $P_+^\vee$ for the corresponding lattices with $\mathbb{Z}_+$ instead of $\mathbb{Z}$. The reflection on V with respect to the hyperplane orthogonal to $\alpha^\vee \in R^\vee$ is defined by

$$
s_{\alpha^\vee}(\mu) := \mu - \langle\alpha^\vee, \mu\rangle\alpha \qquad \text{for} \quad \mu \in V. \tag{2.2}
$$

The reflections associated with the simple roots $\{s_{\alpha^\vee} | \alpha^\vee \in \Pi^\vee\}$, i.e., the simple reflections, generate the Weyl group W. The simple reflections are related to each other by $(s_{\alpha_i^\vee} s_{\alpha_j^\vee})^{m_{ij}} = 1$, where $m_{ij} = 2, 3, 4, 6$ if $\alpha_i^\vee$ and $\alpha_j^\vee$ are connected by 0, 1, 2, 3 laces in the dual Dynkin diagram Γ, respectively. The length l of $w \in W$ is defined from a reduced (shortest) expression $w = s_{j_l} \ldots s_{j_2} s_{j_1}$. We denote the set of distinct weights lying in the W-orbit of $\mu \in P$ by $W(\mu)$ and a unique dominant weight in $W(\mu)$ by $\mu^+ (\in P_+)$. We define the order $\leqslant$ on P by

$$
\nu \leqslant \mu \qquad (\mu, \nu \in P) \Leftrightarrow \mu - \nu \in Q_+.
$$

We turn to the affine coroot system $\hat{R}^\vee := R^\vee \times \mathbb{Z}K \subset \hat{V} := V \oplus \mathbb{R}K$. Let $\hat{R}_+ = \{\alpha^\vee + kK | \alpha \in R^\vee, k > 0\} \cup \{\alpha^\vee \in R_+^\vee\}$ be the set of positive affine coroots and let $\hat{I} = \{0, 1, \ldots, N\}$ be a set of indices. Let $\theta^\vee$ be the highest root in $R^\vee$ and $\hat{\Pi}^\vee = \{\alpha_i^\vee | i \in \hat{I}\}$ the root basis of $\hat{R}^\vee$, where $\alpha_0^\vee := -\theta^\vee + K$. We take a pairing on $\hat{V} \times V$ as $\langle \hat{\lambda}, \mu \rangle = \langle \lambda, \mu \rangle - h$ for $\lambda := \lambda + hK \in \hat{V}$ and $\mu \in V$ and define the fundamental alcove $C = \{\mu \in V | \langle \alpha^\vee, \mu \rangle < 0, \alpha^\vee \in \hat{\Pi}^\vee\}$. The affine reflection on $\hat{V}$ relative to $\hat{\alpha}^\vee = \alpha^\vee + kK \in \hat{R}^\vee$, $(\alpha^\vee \in R^\vee)$ is defined by

$$s_{\hat{\alpha}^\vee}(\hat{\lambda}) := \hat{\lambda} - \langle \lambda, \alpha^\vee \rangle \hat{\alpha}^\vee \qquad \text{for} \quad \hat{\lambda} = \lambda + hK \in \hat{V}$$

which induces the dual action on V through the pairing $\langle \cdot, \cdot \rangle$,

$$s_{\hat{\alpha}^\vee}\langle \mu \rangle = \mu - (\langle \alpha^\vee, \mu \rangle - k)\alpha \qquad \text{for} \quad \mu \in V.$$

The affine reflections associated with the simple roots $\{s_{\alpha^\vee} | \alpha^\vee \in \hat{\Pi}^\vee\}$ generate the affine Weyl group $\hat{W}$. Let τ_ν, $(\nu \in P)$ be the translation on $\hat{V}$,

$$\tau_\nu(\hat{\lambda}) := \hat{\lambda} - \langle \lambda, \nu \rangle K \qquad \text{for} \quad \hat{\lambda} = \lambda + hK \in \hat{V}$$

which induces the translation on V,

$$\tau_\nu\langle \mu \rangle = \mu - \nu \qquad \text{for} \quad \mu \in V.$$

One sees that the elements $\{\tau_\kappa | \kappa \in P^\vee\}$ are mutually commutative. The affine Weyl group $\hat{W}$ contains the element $\tau_\alpha = s_{-\alpha^\vee + K} s_{\alpha^\vee}$, $(\alpha \in Q)$ which is interpreted as a translation corresponding to the root $\alpha \in Q$. In fact, the affine Weyl group is isomorphic to the semidirect product $\hat{W} \simeq W \ltimes \tau_Q$.

We define the extended affine Weyl group by a semidirect product $\tilde{W} := W \ltimes \tau_P$. One finds that $\tilde{W}$ is defined so as to preserve the affine coroot systems $\hat{R}^\vee$. Let $\Omega := \{\tilde{w} \in \tilde{W} | \tilde{w}(C) = C\}$. The extended affine Weyl group is isomorphic to a semidirect product $\tilde{W} \simeq \Omega \ltimes \hat{W}$. Let $\mathcal{O}$ be a set of indices of the image of $\alpha_0^\vee$ by the automorphism of the dual Dynkin diagram Γ. A weight $\mu \in P_+$ satisfying $0 \leqslant \langle \alpha^\vee, \mu \rangle \leqslant 1$ for every $\alpha^\vee \in R_+^\vee$ is called a minuscule weight. It is known that the set of minuscule weights is given by $\{\Lambda_r | r \in \mathcal{O}\}$, where we put $\Lambda_0 = 0$ (see, for example, [5]). One sees a decomposition $\tau_{\Lambda_r} = \omega_r w_r$ with $\omega_r \in \Omega$ and $w_r \in W$ if and only if $r \in \mathcal{O}$. Each ω_r is distinguished by $\omega_r(\alpha_0) = \alpha_r$. Note that $\omega_0 = w_0 = 1$. Let $r^* \in \mathcal{O}$ be the index such that $\alpha_{r^*} = \omega_r^{-1}(\alpha_0)$.

We extend the definition of the length to an element $\tilde{w} \in \tilde{W}$ as

$$l(\tilde{w}) := |R_{\tilde{w}}^\vee| \qquad \text{where} \quad R_{\tilde{w}}^\vee := \hat{R}_+^\vee \cap \tilde{w}^{-1} \hat{R}_-^\vee \tag{2.3}$$

which is consistent with that for $w \in W$, that is $l = l(w)$. Here $R_{\tilde{w}}^\vee$ is the set of positive coroots which become negative coroots by the action of $\tilde{w}$. If we take a reduced expression of $\tilde{w} \in \tilde{W}$ as $\tilde{w} = \omega_r s_{i_l} \ldots s_{i_2} s_{i_1}$, the set $R_{\tilde{w}}^\vee$ is explicitly given by

$$R_{\tilde{w}}^\vee = \{\alpha^{(1)} = \alpha_{i_1}, \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \ldots, \alpha^{(l)} = s_{i_1} s_{i_2} \ldots s_{i_{l-1}}(\alpha_{i_l})\} \tag{2.4}$$

which is independent of the decomposition of $\tilde{w}$. For $\tilde{w} = \tau_\lambda$, we have

$$R_{\tau_\mu}^\vee = \{\alpha^\vee + kK | \alpha^\vee \in R_+^\vee, \langle \alpha^\vee, \mu \rangle > k \geqslant 0\} \cup \{\alpha^\vee + kK | \alpha^\vee \in R_-^\vee, \langle \alpha^\vee, \mu \rangle \geqslant k > 0\}. \tag{2.5}$$

We take a set of parameters $\{k_\alpha \in \mathbb{C} | \alpha \in R\}$ such that $k_\alpha = k_{w(\alpha)}$ for $w \in W$. We write $k_i = k_{\alpha_i}$, $(i \in I)$. Define

$$\rho = \tfrac{1}{2} \sum_{\alpha \in R_+} \alpha \qquad \rho_k = \tfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha. \tag{2.6}$$

Let $P_{++} := P_+ + \rho$ be the regular dominant weight lattice. We see $\langle \alpha^\vee, \mu \rangle \gneq 0$ for $\mu \in P_{++}$ and $\alpha^\vee \in \Pi^\vee$.

2.2. Degenerate double affine Hecke algebras

Following Opdam [26] and Cherednik [4, 6], we introduce degenerate double affine Hecke algebras $\mathcal{DH}$. The commutative elements of $\mathcal{DH}$ give the commutative differential operators in $\mathrm{End}(\mathbb{C}[P])$ which are referred to as the Dunkl–Cherednik operators [4, 7] and uniquely characterize Heckman–Opdam's nonsymmetric Jacobi polynomials as their eigenvectors.

Definition 2.1 (Degenerate double affine Hecke algebra). *The degenerate double affine Hecke algebra $\mathcal{DH}$ is generated over the field $\mathbb{C}$ by the elements $\{s_i, \omega_r, D^{\Lambda_j^\vee} | i \in \hat{I}, r \in \mathcal{O}, j \in I\}$ satisfying*

$$\begin{array}{lll}
(i) & (s_i s_j)^{m_{ij}} = 1 & \text{for} \quad 0 \leqslant i, j \leqslant N \\
(ii) & \omega_r s_i \omega_r^{-1} = s_j \quad \text{if} \quad \omega_r(\alpha_i^\vee) = \alpha_j^\vee & \text{for} \quad r \in \mathcal{O} \\
(iii) & D^\lambda D^\mu = D^\mu D^\lambda & \text{for} \quad \lambda, \mu \in P^\vee \\
(iv) & s_i D^\lambda = D^\lambda s_i \quad \text{if} \quad \langle \lambda, \alpha_i \rangle = 0 & \text{for} \quad 1 \leqslant i \leqslant N \\
 & s_0 D^\lambda = D^\lambda s_0 \quad \text{if} \quad \langle \lambda, -\theta \rangle = 0 & \\
(v) & s_i D^\lambda - D^{s_i(\lambda)} s_i = k_i \quad \text{if} \quad \langle \lambda, \alpha_i \rangle = 1 & \text{for} \quad 1 \leqslant i \leqslant N \\
 & s_0 D^\lambda - D^{s_0(\lambda)} s_0 = k_\theta \quad \text{if} \quad \langle \lambda, -\theta \rangle = 1 & \\
(vi) & \omega_r D^\lambda \omega_r^{-1} = D^{\omega_r(\lambda)} = D^{w_r^{-1}(\lambda)} - \langle \lambda, \Lambda_{r^*} \rangle & \text{for} \quad r \in \mathcal{O}
\end{array} \tag{2.7}$$

where

$$D^{\hat{\lambda}} = \sum_{i \in I} \lambda_i D^{\Lambda_i^\vee} - h \qquad \text{for} \quad \hat{\lambda} = \sum_{i \in I} \lambda_i \Lambda_i^\vee + hK.$$

The degenerate double affine Hecke algebra $\mathcal{DH}$ contains the extended affine Weyl group $\tilde{W}$ which acts on the affine coroot system $\hat{R}^\vee$. Repeated use of the defining relations of $\mathcal{DH}$ gives the relations

$$\begin{array}{ll}
s_i D^\lambda - D^{s_i(\lambda)} s_i = k_i \langle \lambda, \alpha_i \rangle & \text{for} \quad 1 \leqslant i \leqslant N \\
s_0 D^\lambda - D^{s_0(\lambda)} s_0 = k_\theta \langle \lambda, -\theta \rangle. &
\end{array} \tag{2.8}$$

We define the following elements:

$$X^{-\mu} := \tau_\mu \in \mathcal{DH} \qquad \text{for} \quad \mu \in P$$

which provide $s_0 = X^{-\theta} s_{\theta^\vee}$ and $\omega_r = X^{-\Lambda_r} w_r^{-1}$, $(r \in \mathcal{O})$.

We introduce the commutative differential operators $\{\hat{D}^{\hat{\lambda}} \in \mathrm{End}(\mathbb{C}[P]) | \hat{\lambda} = \lambda + hK \in \hat{P}^\vee\}$,

$$\hat{D}^{\hat{\lambda}} f := \partial^\lambda(f) + \sum_{\alpha \in R_+} \frac{k_\alpha \langle \lambda, \alpha \rangle}{x^\alpha - 1} (f - \hat{s}_\alpha(f)) + \langle \lambda, \rho_k \rangle f - hf \tag{2.9}$$

where $\{\partial^\lambda \in \mathrm{End}(\mathbb{C}[P]) | \lambda \in P^\vee\}$ is the derivative of $\mathbb{C}[P]$:

$$\partial^\lambda(x^\mu) = \langle \lambda, \mu \rangle x^\mu \qquad \text{for} \quad x^\mu \in \mathbb{C}[P]$$

and the elements $w \in W$ act on $\mathbb{C}[P]$ as

$$w(x^\mu) = x^{w(\mu)} \qquad \text{for} \quad x^\mu \in \mathbb{C}[P].$$

We consider x^μ, $(\mu \in P)$ as operators of multiplication of x^μ. The differential operators $\{\hat{D}^\lambda | \lambda \in P^\vee\}$ are called the Dunkl–Cherednik operators [4, 7]. One sees that a map $\pi : \mathcal{DH} \to \mathrm{End}(\mathbb{C}[P])$ defined by

$$\pi : s_i \mapsto s_i, (1 \leqslant i \leqslant N) \qquad D^{\hat{\lambda}} \mapsto \hat{D}^{\hat{\lambda}} \qquad X^\mu \mapsto x^\mu \tag{2.10}$$

gives a faithful representation of $\mathcal{DH}$. The Dunkl–Cherednik operators $\{\hat{D}^\lambda | \lambda \in P^\vee\}$ have the triangularity in $\mathbb{C}[P]$:

$$\hat{D}^\lambda x^\mu = \langle \lambda, \mu + \rho_k(\mu) \rangle x^\mu + \sum_{\nu \prec \mu} c_{\mu\nu} x^\nu \qquad \mu \in P \qquad c_{\mu\nu} \in \mathbb{K} \tag{2.11}$$

where we denote by w_μ the shortest element of W such that $w_\mu^{-1}(\mu) \in P_+$ and define $\rho_k(\mu) := w_\mu(\rho_k)$. The order $\preceq$ on P is defined by

$$\nu \preceq \mu, (\mu, \nu \in P) \Leftrightarrow \begin{cases} \text{if} \quad \mu^+ \neq \nu^+ & \text{then} \quad \nu^+ < \mu^+ \\ \text{if} \quad \mu^+ = \nu^+ & \text{then} \quad \nu \leqslant \mu. \end{cases} \tag{2.12}$$

3. Nonsymmetric Heckman–Opdam polynomials

We investigate the eigenvectors of the Dunkl–Cherednik operators $\{\hat{D}^\lambda | \lambda \in P^\vee\}$. Due to the triangularity (2.11), there exists a family of polynomials $F_\mu := F_\mu(x; \{k_\alpha\}) \in \mathbb{C}[P]$, $(\mu \in P)$ satisfying the following conditions:

$$\begin{aligned} &(i) \quad F_\mu = x^\mu + \sum_{\nu \prec \mu} w_{\mu\nu} x^\nu \qquad w_{\mu\nu} \in \mathbb{C} \\ &(ii) \quad \hat{D}^\lambda F_\mu = \langle \lambda, \mu + \rho_k(\mu) \rangle F_\mu. \end{aligned} \tag{3.1}$$

F_μ are the nonsymmetric Jacobi polynomials introduced by Heckman and Opdam. Hereafter, we call them the nonsymmetric Heckman–Opdam polynomials. Note that all the eigenspaces of the Dunkl–Cherednik operators $\{\hat{D}^\lambda | \lambda \in P^\vee\}$ are one-dimensional. Applying $\{s_i \in \mathrm{End}(\mathbb{C}[P]) | i \in I\}$ to the nonsymmetric Heckman–Opdam polynomials F_μ, $(\mu \in P)$, we see that

$$s_i F_\mu = \begin{cases} \dfrac{k_i}{\langle \alpha_i^\vee, \mu + \rho_k(\mu) \rangle} F_\mu + F_{s_i(\mu)} & \text{if} \quad \langle \alpha_i^\vee, \mu \rangle < 0 \\ F_\mu & \text{if} \quad \langle \alpha_i^\vee, \mu \rangle = 0 \\ \dfrac{k_i}{\langle \alpha_i^\vee, \mu + \rho_k(\mu) \rangle} F_\mu + \dfrac{\langle \alpha_i^\vee, \mu + \rho_k(\mu) \rangle^2 - k_i^2}{\langle \alpha_i^\vee, \mu + \rho_k(\mu) \rangle^2} F_{s_i(\mu)} & \text{if} \quad \langle \alpha_i^\vee, \mu \rangle > 0. \end{cases} \tag{3.2}$$

We introduce intertwiners in $\mathcal{DH}$ to provide the Rodrigues formulae for the nonsymmetric Heckman–Opdam polynomials. These intertwiners were first considered in the context of the Hecke algebras. The intertwiners in $\mathcal{DH}$ were developed by Cherednik [6, 28]. With their use, Opdam derived the evaluation formulae of the nonsymmetric Heckman–Opdam polynomials [26]. We construct the Rodrigues formulae for the nonsymmetric Heckman–Opdam polynomials and algebraically evaluate their square norms.

Definition 3.1. *(i) We define $\{K_i \in \mathcal{DH} | i \in \hat{I}\}$ by*

$$K_0 := s_\theta X^\theta D^{\alpha_0^\vee} - k_\theta \qquad K_i := s_i D^{\alpha_i^\vee} - k_i \tag{3.3}$$

which we call the intertwiners.

(ii) For a reduced expression $w = \omega_r s_{i_l} \ldots s_{i_2} s_{i_1} \in \tilde{W}$, we define $K_w := \omega_r K_{i_l} \ldots K_{i_2} K_{i_1} (\in \mathcal{DH})$. In particular, we write $B_\mu := K_{\tau_{-\mu}}$ for $\mu \in P_+$. We call $\{\hat{B}_\mu := \pi(B_\mu) \in \mathrm{End}(\mathbb{C}[P])\}$ the raising operators.

The elements $\{K_w | w \in \tilde{W}\}$ have the following relations:

$$\begin{aligned} &(i) \quad K_i K_j K_i \ldots = K_j K_i K_j \ldots, m_{ij} \text{ factors on each side} \\ &\qquad \omega_r K_i \omega_r^{-1} = K_j \qquad \text{if} \quad \omega_r(\alpha_i^\vee) = \alpha_j^\vee \\ &(ii) \quad K_i^2 = -(D^{\alpha_i^\vee})^2 + k_i^2 \qquad \text{for} \quad i \in \hat{I} \\ &(iii) \quad K_w D^\lambda = D^{w(\lambda)} K_w \qquad \text{for} \quad \lambda \in P^\vee. \end{aligned} \tag{3.4}$$

The first relations in (3.4) are called the braid relations. The last relations give the reason why they are called intertwiners. For the elements $\{B_\mu \in \mathcal{DH}\}$, we can show that

$$B_\mu D^\lambda = D^{\tau_{-\mu}(\lambda)} B_\mu = (D^\mu - \langle \lambda, \mu \rangle) B_\mu. \tag{3.5}$$

By applying the raising operators $\hat{B}_\mu$ to the nonsymmetric Heckman–Opdam polynomials F_ν, $(\nu \in P_+)$, we obtain

$$\hat{D}^\lambda(\hat{B}_\mu F_\nu) = \hat{B}_\mu(\hat{D}^\lambda + \langle \lambda, \mu \rangle) F_\nu = \langle \lambda, \mu + \nu + \rho_k \rangle (\hat{B}_\mu F_\nu). \tag{3.6}$$

Hence $\hat{B}_\mu F_\nu$ coincides with $F_{\mu+\nu}$ up to a constant factor. If we apply $\hat{K}_w := \pi(K_w)$, $(w \in W)$ to F_μ, $(\mu \in P_+)$, we see that

$$\hat{D}^\lambda(\hat{K}_w F_\mu) = \hat{K}_w \hat{D}^{w^{-1}(\lambda)} F_\mu = \langle w^{-1}(\lambda), \mu + \rho_k \rangle \hat{K}_w F_\mu = \langle \lambda, w(\mu + \rho_k) \rangle (\hat{K}_w F_\mu). \tag{3.7}$$

Hence $\hat{K}_w F_\mu$ coincides with $F_{w(\mu)}$ up to a constant factor.

Theorem 3.2 (Rodrigues formulae). *(i) For a dominant weight $\mu \in P_+$, we construct the nonsymmetric Heckman–Opdam polynomials F_μ by applying the raising operators $\{\hat{B}_\mu | \mu \in P_+\}$ to the reference state $F_0 = 1$,*

$$F_\mu = c_\mu^{-1} \hat{B}_\mu F_0 \tag{3.8}$$

where the coefficient of the top term is given by

$$c_\mu = \prod_{\alpha^\vee \in R^\vee_{\tau_{-\mu}}} \langle \alpha^\vee, \rho_k \rangle.$$

(ii) For a general weight $\mu \in P$, we construct the nonsymmetric Heckman–Opdam polynomials F_μ by applying the operator $\hat{K}_{w_\mu}$, $(w_\mu \in W, w_\mu^{-1}(\mu) =: \mu^+ \in P_+)$ to F_{μ^+},

$$F_\mu = c_{w_\mu}^{-1} \hat{K}_{w_\mu} F_{\mu^+} \tag{3.9}$$

where the coefficient of the top term is

$$c_{w_\mu} = \prod_{\alpha^\vee \in R^\vee_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + \rho_k \rangle^2 - k_\alpha^2}{\langle \alpha^\vee, \mu^+ + \rho_k \rangle}.$$

See our previous paper [23] for the detailed proofs for the coefficients of the top terms appearing in the Rodrigues formulae.

In the remainder of this paper, we assume $k_\alpha \geqslant 0$, $(\alpha \in R)$. Define the inner product $\langle \cdot, \cdot \rangle_k$ by

$$\langle f, g \rangle_k := \int_T f(t) \overline{g(t)} \Delta_k(t) \, \mathrm{d}\mu \tag{3.10}$$

where $T = V/2\pi Q^\vee$ is a torus, $x^\mu(t) := \mathrm{e}^{\sqrt{-1}\langle t, \mu \rangle}$, $(t \in T)$, $\mathrm{d}\mu$ is the normalized Haar measure on T and the weight function Δ_k is given by

$$\Delta_k := \prod_{\alpha \in R} |1 - x^\alpha|^{k_\alpha}. \tag{3.11}$$

The square norm of the reference state is given by

$$\langle 1, 1 \rangle_k = \prod_{\alpha \in R_+} \frac{\Gamma(\langle \alpha^\vee, \rho_k \rangle + k_\alpha + 1)\Gamma(\langle \alpha^\vee, \rho_k \rangle - k_\alpha + 1)}{\Gamma(\langle \alpha^\vee, \rho_k \rangle + 1)^2} \tag{3.12}$$

which is indeed evaluated from Opdam's shift operators [25]. Since the Dunkl–Cherednik operators $\{\hat{D}^{\lambda}\}$ are selfadjoint with respect to the inner product $\langle\cdot,\cdot\rangle_k$ (3.10), we have the orthogonality

$$\langle F_\mu, F_\nu\rangle_k = 0 \qquad \text{if} \quad \mu \neq \nu. \tag{3.13}$$

In fact, the nonsymmetric Heckman–Opdam polynomials form an orthogonal basis in $\mathbb{C}[P]$ with respect to the inner product $\langle\cdot,\cdot\rangle_k$ (3.10). We see that the adjoint operators of $\{\omega_r, \hat{K}_i | r \in O, i \in \hat{I}\}$ are given by

$$\omega_r^* = \omega_r^{-1} \qquad \hat{K}_i^* = -\hat{K}_i. \tag{3.14}$$

Theorem 3.3. *For a dominant weight $\mu \in P_+$, we have*

$$\langle F_\mu, F_\mu\rangle_k = \prod_{\alpha\in R_+} \frac{\Gamma(\langle\alpha^\vee, \mu+\rho_k\rangle + k_\alpha + 1)\Gamma(\langle\alpha^\vee, \mu+\rho_k\rangle - k_\alpha + 1)}{\Gamma(\langle\alpha^\vee, \mu+\rho_k\rangle + 1)^2}. \tag{3.15}$$

Proof. Define $N(\hat{\alpha}^\vee) \in \mathcal{DH}$, $(\hat{\alpha}^\vee = \alpha^\vee + kK \in \hat{R}^\vee)$ by

$$N(\hat{\alpha}^\vee) := (D^{\alpha^\vee})^2 - k_\alpha^2. \tag{3.16}$$

Since they satisfy the following properties:

$$N(\hat{\alpha}_i^\vee) = -K_i^2 \qquad K_w N(\hat{\alpha}^\vee) = N(w(\hat{\alpha}^\vee))K_w \qquad \text{for} \quad w \in \tilde{W}$$

the product $\hat{B}_\mu^* \hat{B}_\mu \in \mathrm{End}(\mathbb{C}[P])$ is written as

$$\begin{aligned}
\hat{B}_\mu^* \hat{B}_\mu &= \omega_r \hat{K}_{i_l}^* \ldots \hat{K}_{i_1}^* \hat{K}_{i_1} \ldots \hat{K}_{i_l} \omega_r^* \\
&= \omega_r \hat{K}_{i_l}^* \ldots \hat{K}_{i_2}^* \hat{N}(\alpha_{i_1}^\vee) \hat{K}_{i_2} \ldots \hat{K}_{i_l} \omega_r^{-1} \\
&= \prod_{\alpha^\vee \in R^\vee_{\tau_{-\mu}}} \hat{N}(\alpha^\vee)
\end{aligned}$$

where $\hat{N}(\alpha^\vee) := \pi(N(\alpha^\vee))$. The square norms of F_μ are calculated as

$$\begin{aligned}
\langle F_\mu, F_\mu\rangle_k &= \langle c_\mu^{-1}\hat{B}_\mu F_0, c_\mu^{-1}\hat{B}_\mu F_0\rangle_k \\
&= (c_\mu)^{-2}\langle F_0, \hat{B}_\mu^* \hat{B}_\mu F_0\rangle_k \\
&= (c_\mu)^{-2} \prod_{\alpha^\vee \in R^\vee_{\tau_{-\mu}}} \langle F_0, \hat{N}(\alpha^\vee) F_0\rangle_k \\
&= \langle F_0, F_0\rangle_k \prod_{\alpha^\vee \in R^\vee_{\tau_{-\mu}}} \frac{\langle\alpha^\vee, \rho_k\rangle^2 - k_\alpha^2}{\langle\alpha^\vee, \rho_k\rangle^2} \\
&= \langle 1, 1\rangle_k \prod_{\alpha\in R_+} \prod_{i=1}^{\langle\alpha^\vee,\mu\rangle} \frac{(\langle\alpha^\vee, \rho_k\rangle + k + i)(\langle\alpha^\vee, \rho_k\rangle - k + i)}{(\langle\alpha^\vee, \rho_k\rangle + i)^2}.
\end{aligned}$$

□

Proposition 3.4. *For a weight $\mu \in P$ lying in the W-orbit of $\mu^+ \in P_+$, we have*

$$\frac{\langle F_\mu, F_\mu\rangle_k}{\langle F_{\mu^+}, F_{\mu^+}\rangle_k} = \prod_{\alpha^\vee \in R^\vee_{w_\mu}} \frac{\langle\alpha^\vee, \mu^+ + \rho_k\rangle^2}{\langle\alpha^\vee, \mu^+ + \rho_k\rangle^2 - k_\alpha^2}. \tag{3.17}$$

Proof. For a reduced expression $w_\mu = s_{i_l} s_{i_2} \dots s_{i_1}$, we have

$$\begin{aligned}\hat{K}^*_{w_\mu}\hat{K}_{w_\mu} &= \hat{K}^*_{i_l}\dots\hat{K}^*_{i_2}\hat{K}^*_{i_1}\hat{K}_{i_1}\hat{K}_{i_2}\dots\hat{K}_{i_l}\\ &= \hat{K}^*_{i_l}\dots\hat{K}^*_{i_2}\hat{N}(\alpha^\vee_{i_1})\hat{K}_{i_2}\dots\hat{K}_{i_l}\\ &= \prod_{\alpha^\vee\in R^\vee_{w_\mu}}\hat{N}(\alpha^\vee).\end{aligned}$$

From the Rodrigues formula (3.9), we calculate the scalar product as follows:

$$\begin{aligned}\langle F_\mu, F_\mu\rangle_k &= \langle c^{-1}_{w_\mu}\hat{K}_{w_\mu}F_{\mu^+}, c^{-1}_{w_\mu}\hat{K}_{w_\mu}F_{\mu^+}\rangle_k\\ &= (c_{w_\mu})^{-2}\langle F_{\mu^+}, \hat{K}^*_{w_\mu}\hat{K}_{w_\mu}F_{\mu^+}\rangle_k\\ &= (c_{w_\mu})^{-2}\prod_{\alpha^\vee\in R^\vee_{w_\mu}}\langle F_{\mu^+}, \hat{N}(\alpha^\vee)F_{\mu^+}\rangle_k\\ &= (c_{w_\mu})^{-2}\langle F_{\mu^+}, F_{\mu^+}\rangle_k\prod_{\alpha^\vee\in R^\vee_{w_\mu}}(\langle\alpha^\vee, \mu^+ + \rho_k\rangle^2 - k^2_\alpha).\end{aligned}$$

□

4. Bosonic and fermionic eigenstates for generalized Sutherland models

The generalized Sutherland models associated with arbitrary reduced root systems are given by

$$H_{\mathrm{S}} := \sum_{i\in I}\partial^{\Lambda^\vee_i}\partial^{\alpha_i} - \sum_{\alpha\in R_+}\frac{\langle\alpha,\alpha\rangle}{(x^{\alpha/2} - x^{-\alpha/2})^2}k_\alpha(k_\alpha - s_\alpha). \tag{4.1}$$

Using the variables $\{t \in T\}$, this is rewritten as

$$H_{\mathrm{S}}(t) = -\triangle + \frac{1}{4}\sum_{\alpha\in R_+}\frac{\langle\alpha,\alpha\rangle}{\sin^2(\langle t,\alpha\rangle/2)}k_\alpha(k_\alpha - s_\alpha) \tag{4.2}$$

where $\triangle$ is the Laplacian on T. The Hamiltonian has so-called exchange terms [1]. If we consider the bosonic (or fermionic) eigenstates, i.e., we restrict the operand of H_{S} to the W-symmetric (or W-anti-asymmetric) function space, we can replace the exchange terms by $s_\alpha = 1$ (or -1),

$$H^{(B,F)}_{\mathrm{S}} = \sum_{i\in I}\partial^{\Lambda^\vee_i}\partial^{\alpha_i} - \sum_{\alpha\in R_+}\frac{\langle\alpha,\alpha\rangle}{(x^{\alpha/2} - x^{-\alpha/2})^2}k_\alpha(k_\alpha \mp 1). \tag{4.3}$$

Through the similarity transformation by a W-symmetric function

$$\phi_k := \prod_{\alpha\in R}|1 - x^\alpha|^{k_\alpha/2} \tag{4.4}$$

we obtain

$$\phi^{-1}_k \circ H_{\mathrm{S}} \circ \phi_k = \sum_{i\in I}\hat{D}^{\Lambda^\vee_i}\hat{D}^{\alpha_i}. \tag{4.5}$$

Hence we see that H_{S} has the eigenvectors in $\mathbb{C}[P]\phi_k$ written by products of the nonsymmetric Heckman–Opdam polynomial F_μ and ϕ_k. We note that ϕ_k corresponds to the ground state wavefunction for the bosonic Hamiltonian $H^{(B)}_{\mathrm{S}}$. Since the nonsymmetric eigenstates expressed by products of ϕ_k and the nonsymmetric Heckman–Opdam polynomials with weights lying in the same W-orbit have the same eigenvalues of H_{S},

$$\begin{aligned} H_{\mathrm{S}}(\phi_k F_\mu) &= \sum_{i\in I} \langle \Lambda_i^\vee, \mu + \rho_k(\mu)\rangle \langle \alpha_i, \mu + \rho_k(\mu)\rangle (\phi_k F_\mu) \\ &= \langle \mu + \rho_k(\mu), \mu + \rho_k(\mu)\rangle (\phi_k F_\mu) \\ &= \langle \mu^+ + \rho_k, \mu^+ + \rho_k\rangle (\phi_k F_\mu) \end{aligned} \tag{4.6}$$

we can take any linear combinations of $\phi_k F_\mu$ with weights lying in a W-orbit as eigenvectors of H_{S} (see [1] for type A). In what follows, we construct the bosonic and fermionic eigenstates of the generalized Sutherland models $H_{\mathrm{S}}^{(B,F)}$ from the W-symmetrized and W-anti-symmetrized nonsymmetric Heckman–Opdam polynomials, respectively.

Theorem 4.1. *Let J_μ^+, $(\mu \in P_+)$ and J_μ^-, $(\mu \in P_{++})$ be the following linear combinations of the nonsymmetric Heckman–Opdam polynomials $F_{\tilde{\mu}}$, $(\tilde{\mu} \in W(\mu))$:*

$$J_\mu^\pm = \sum_{\tilde{\mu}\in W(\mu)} b_{\mu\tilde{\mu}}^\pm F_{\tilde{\mu}} \tag{4.7}$$

where

$$b_{\mu\tilde{\mu}}^\pm = \prod_{\alpha^\vee \in R_{w_{\tilde{\mu}}}^\vee} \pm \frac{\langle \alpha^\vee, \mu + \rho_k\rangle \mp k_\alpha}{\langle \alpha^\vee, \mu + \rho_k\rangle}. \tag{4.8}$$

The polynomials J_μ^+, $(\mu \in P_+)$ and J_μ^-, $(\mu \in P_{++})$ are elements of $\mathbb{C}[P]^{\pm W}$ and called the symmetric and the anti-symmetric Heckman–Opdam polynomials respectively.

It is sufficient to require the conditions $s_i J_\mu^\pm = \pm J_\mu^\pm$ and $b_{\mu\mu}^\pm = 1$ in order to determine the coefficients $b_{\mu\tilde{\mu}}^\pm$ such that $J_\mu^\pm \in \mathbb{C}[P]^{\pm W}$.

The symmetric Heckman–Opdam polynomials of type A are equivalent to the (symmetric) Jack polynomials. Symmetrization of the nonsymmetric Jack polynomials was carried out by Baker and Forrester [2]. We remark that their method with arm- and leg-lengths of the Young diagram is essentially different from our approach.

We obtain the Rodrigues formulae for the symmetric and the anti-symmetric Heckman–Opdam polynomials $J_\mu^\pm$, ($\mu \in P_+$ for J_μ^+ and $\mu \in P_{++}$ for J_μ^-),

$$J_\mu^\pm = \sum_{\tilde{\mu}\in W(\mu)} b_{\mu\tilde{\mu}}^\pm c_{w_{\tilde{\mu}}}^{-1} c_\mu^{-1} \hat{K}_{w_{\tilde{\mu}}} \hat{B}_\mu F_0. \tag{4.9}$$

As a result, we find the bosonic and the fermionic eigenstates $\phi_\mu^{(B,F)}$, ($\mu \in P_+$ for $\phi_\mu^{(B)}$ and $\mu \in P_{++}$ for $\phi_\mu^{(F)}$) for the generalized Sutherland models $H_{\mathrm{S}}^{(B,F)}$,

$$\begin{aligned} \phi_\mu^{(B,F)} &= \phi_k \sum_{\tilde{\mu}\in W(\mu)} b_{\mu\tilde{\mu}}^\pm c_{w_{\tilde{\mu}}}^{-1} c_\mu^{-1} \hat{K}_{w_{\tilde{\mu}}} \hat{B}_\mu F_0 \\ H_{\mathrm{S}}^{(B,F)} \phi_\mu^{(B,F)} &= \langle \mu + \rho_k, \mu + \rho_k\rangle \phi_\mu^{(B,F)} \end{aligned} \tag{4.10}$$

respectively. Lapointe and Vinet obtained the Rodrigues formulae for the Jack polynomials [15]. The relation between our formulae (4.9) and theirs has not been clarified.

We proceed to the evaluation of square norms of the eigenstates $\phi^{(B,F)}$,

$$\|\phi_\mu^{(B,F)}\|^2 = \int_T |\phi_\mu^{(B,F)}(t)|^2 \, \mathrm{d}\mu = \langle J_\mu^\pm, J_\mu^\pm\rangle_k. \tag{4.11}$$

To prove a theorem, we need the following lemma.

Lemma 4.2. *For $\mu \in P_+$, we have an identity,*

$$\sum_{\tilde{\mu}\in W(\mu)} \prod_{\alpha^\vee \in R^\vee_{w_{\tilde{\mu}}}} \frac{\langle \alpha^\vee, \mu + \rho_k\rangle \mp k_\alpha}{\langle \alpha^\vee, \mu + \rho_k\rangle \pm k_\alpha} = \prod_{\alpha\in R_+} \frac{\langle \alpha^\vee, \mu + \rho_k\rangle}{\langle \alpha^\vee, \mu + \rho_k\rangle \pm k_\alpha}. \tag{4.12}$$

The identity is proved by using an expression of the Poincaré polynomials [17, 23]. We show a proof in the appendix.

Theorem 4.3. *Let $\mu \in P_+$ for J^+_μ and let $\mu \in P_{++}$ for J^-_μ. We have*

$$\langle J^\pm_\mu, J^\pm_\nu\rangle_k = \delta_{\mu\nu} \prod_{\alpha\in R_+} \frac{\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle \pm k_\alpha)\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle \mp k_\alpha + 1)}{\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle)\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle + 1)}. \tag{4.13}$$

Proof. The orthogonality for $\mu \neq \nu$ is straightforward from that of the nonsymmetric Heckman–Opdam polynomials (3.13). We have

$$\begin{aligned}
\langle J^\pm_\mu, J^\pm_\mu\rangle_k &= \sum_{\tilde{\mu}\in W(\mu)} (b^\pm_{\mu\tilde{\mu}})^2 \langle F_{\tilde{\mu}}, F_{\tilde{\mu}}\rangle_k \\
&= \sum_{\tilde{\mu}\in W(\mu)} (b^\pm_{\mu\tilde{\mu}})^2 \frac{\langle F_{\tilde{\mu}}, F_{\tilde{\mu}}\rangle_k}{\langle F_\mu, F_\mu\rangle_k} \langle F_\mu, F_\mu\rangle_k \\
&= \sum_{\tilde{\mu}\in W(\mu)} \prod_{\alpha^\vee \in R^\vee_{w_{\tilde{\mu}}}} \frac{\langle \alpha^\vee, \mu + \rho_k\rangle \mp k_\alpha}{\langle \alpha^\vee, \mu + \rho_k\rangle \pm k_\alpha} \langle F_\mu, F_\mu\rangle_k \\
&= \prod_{\alpha\in R_+} \frac{\langle \alpha^\vee, \mu + \rho_k\rangle}{\langle \alpha^\vee, \mu + \rho_k\rangle \pm k_\alpha} \langle F_\mu, F_\mu\rangle_k
\end{aligned}$$

where the last equality follows from lemma 4.2. □

Corollary 4.4. *For $k_\alpha \in \mathbb{N}$, we have*

$$\langle J^\pm_\mu, J^\pm_\nu\rangle_k = \delta_{\mu,\nu} \prod_{\alpha\in R_+} \prod_{i=1}^{k_\alpha - 1} \frac{\langle \alpha^\vee, \mu + \rho_k\rangle \pm i}{\langle \alpha^\vee, \mu + \rho_k\rangle \mp i}. \tag{4.14}$$

We remark that the inner products (4.14) were first proved by use of Opdam's shift operators [25]. From (4.11) and (4.12), we obtain square norms of the eigenstates $\phi^{(B,F)}_\mu$ for the generalized Sutherland models,

$$\|\phi^{(B,F)}_\mu\|^2 = \prod_{\alpha\in R_+} \frac{\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle \pm k_\alpha)\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle \mp k_\alpha + 1)}{\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle)\Gamma(\langle \alpha^\vee, \mu + \rho_k\rangle + 1)}. \tag{4.15}$$

5. Concluding remarks

We summarize the results in this paper. First, we have presented the Rodrigues formulae for the nonsymmetric Heckman–Opdam polynomials which correspond to the nonsymmetric basis of the generalized Sutherland models with exchange terms. Their square norms are evaluated in an algebraic manner. Second, through W-symmetrization and W-anti-symmetrization of the nonsymmetric Heckman–Opdam polynomials we have constructed the bosonic and fermionic eigenstates of the generalized Sutherland models with arbitrary reduced root systems, respectively. The square norms of the eigenstates are calculated from their nonsymmetric counterparts through an expression of the Poincaré polynomials.

We consider some interesting applications and extensions of our method. The generalized Sutherland models we have studied in this paper do not include the models associated with the BC_N-type nonreduced root system. Since we have already obtained the Rodrigues formulae for the nonsymmetric Heckman–Opdam polynomials of type BC_N [22, 27, 34], the extension should be straightforward. And it is known that there exists the symmetric orthogonal basis for the Calogero model which describes many particles with inverse-square interactions in a harmonic well [20, 32–34]. We have already confirmed that our method can be applied to the eigenstates of the Calogero model. The detail of the analysis will be reported elsewhere.

Acknowledgments

The authors would like to thank Dr H Ujino and Dr Y Komori for valuable discussions. AN would like to express his gratitude to Professor M J Ablowitz, Professor H Segur, Professor L Vinet and their colleagues in Department of Applied Mathematics, University of Colorado and Centre de Recherches Mathématique, Université de Montréal for giving kind hospitality and helpful comments during his stay in Boulder and Montreal.

Appendix. Proof of lemma 4.2

We present a proof of lemma 4.2 following our previous paper [23].

The Poincaré polynomials associated with the Weyl group [10] are given by

$$W(t) = \sum_{w\in W} \prod_{\alpha\in R_w} t_\alpha \tag{A.1}$$

where $\{t_\alpha | \alpha \in R\}$ are W-invariant indeterminates, i.e., $t_\alpha = t_{w(\alpha)}$ for $w \in W$. We denote by $\mathbb{K}$ the field of rational functions over $\mathbb{C}$ in square-roots of indeterminates $\{t_\alpha\}$. To investigate the Poincaré polynomials, Macdonald proved the following identity [17].

Theorem A.1 (I G Macdonald).

$$W(t) = \sum_{w\in W} \prod_{\alpha\in R_+} \frac{1 - t_\alpha x^{w(\alpha^\vee)}}{1 - x^{w(\alpha^\vee)}}. \tag{A.2}$$

Lemma A.2. *Let $\mu \in P_+$. We have*

$$\sum_{\tilde{\mu}\in W(\mu)} \prod_{\alpha^\vee\in R^\vee_{w_{\tilde{\mu}}}} \frac{t_\alpha(1 - t_\alpha^{-1} q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle})}{1 - t_\alpha q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}} = W(t) \prod_{\alpha\in R_+} \frac{1 - q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}}{1 - t_\alpha q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}}. \tag{A.3}$$

Proof. There exists a $\mathbb{K}$-homomorphism $\varphi : \mathbb{K}[Q^\vee] \to \mathbb{K}$ defined by

$$\varphi : x^{\alpha_i^\vee} \mapsto q^{\pm\langle\alpha_i^\vee,\mu+\rho_k\rangle} \qquad \text{for} \quad i \in I.$$

Since $W(t) \in \mathbb{K}[Q^\vee]$ does not depend on $\{x^{\alpha_i^\vee}\}$ as (A.2), we have

$$\begin{aligned}
\varphi(W(t)) &= W(t) \\
&= \sum_{w\in W} \prod_{\alpha\in R_+} \varphi\left(\frac{1 - t_\alpha x^{w(\alpha^\vee)}}{1 - x^{w(\alpha^\vee)}}\right) \\
&= \sum_{w\in W} \prod_{\alpha\in R_+} \frac{1 - t_\alpha q^{\pm\langle w(\alpha^\vee),\mu+\rho_k\rangle}}{1 - q^{\pm\langle w(\alpha^\vee),\mu+\rho_k\rangle}} \\
&= \frac{\sum_{w\in W} \prod_{\alpha^\vee\in R^\vee_w} (t_\alpha - q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}) \prod_{\alpha^\vee\in R^\vee_+\setminus R^\vee_w} (1 - t_\alpha q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle})}{\prod_{\alpha\in R_+} (1 - q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle})}.
\end{aligned}$$

Thus we obtain the following relation:

$$\sum_{w\in W}\prod_{\alpha^\vee\in R_w^\vee}\frac{t_\alpha(1-t_\alpha^{-1}q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle})}{1-t_\alpha q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}} = W(t)\prod_{\alpha\in R_+}\frac{1-q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}}{1-t_\alpha q^{\pm\langle\alpha^\vee,\mu+\rho_k\rangle}}. \tag{A.4}$$

We show that the sum on the left-hand side of the above equation can be replaced by the sum on $\tilde{\mu}\in W(\mu)$. Consider the isotropy group $W_\mu=\{w\in W|w(\mu)=\mu\}$ for the dominant weight $\mu\in P_+$ ($W_\mu=\{1\}$ for $\mu\in P_{++}$). Since an element $w\in W_\mu\setminus\{1\}$ can be written by a product of simple reflections fixing μ, $\{s_i|i\in J\subset I\}$(see [10]), there exists at least one simple root $\alpha_i^\vee\in\Pi^\vee$ associated with the reflection s_i in the set $R_w^\vee$. Hence, for $w\in W_\mu\setminus\{1\}$, we have

$$\begin{aligned}\prod_{\alpha^\vee\in R_w^\vee}t_\alpha(1-t_\alpha^{-1}q^{\langle\alpha^\vee,\mu+\rho_k\rangle}) &= t_i(1-t_i^{-1}q^{\langle\alpha_i^\vee,\rho_k\rangle})\prod_{\alpha^\vee\in R_w^\vee\setminus\{\alpha_i^\vee\}}t_\alpha(1-t_\alpha^{-1}q^{\langle\alpha^\vee,\mu+\rho_k\rangle})\\ &= t_i(1-t_i^{-1}t_i)\prod_{\alpha^\vee\in R_w^\vee\setminus\{\alpha_i^\vee\}}t_\alpha(1-t_\alpha^{-1}q^{\langle\alpha^\vee,\mu+\rho_k\rangle})=0.\end{aligned}$$

Define $W^\mu := \{w\in W|l(ws_i)>l(w)$ for all $i\in J\}$. For $w\in W$, there is a unique $u\in W^\mu$ and a unique $v\in W_\mu$ such that $w=uv$. We obtain the above lemma since the sum on $w\in W$ on the left-hand side of (A.4) can be replaced by that on $w\in W^\mu$ which is equivalent to that on $\tilde{\mu}\in W(\mu)$. □

In the formal limit $q\to 1$ under the restriction $t_\alpha=q^{k_\alpha}$, we have relation (4.12) in lemma 4.2.

References

[1] Baker T H and Forrester P J 1997 The Calogero–Sutherland model and polynomials with prescribed symmetry *Nucl. Phys.* B **492** 682–716

[2] Baker T H and Forrester P J 1997 Symmetric Jack polynomials from non-symmetric theory *Preprint* q-alg/9707001

[3] Calogero F 1971 Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419–36

[4] Cherednik I 1991 A unification of the Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras *Invent. Math.* **106** 411–32

[5] Cherednik I 1995 Double-affine Hecke algebras and Macdonald's conjectures *Ann. Math.* **95** 191–216

[6] Cherednik I 1997 Intertwining operators of double-affine Hecke algebras *Selecta Math.* **3** 459–95

[7] Dunkl C F 1989 Differential-difference operators associated to reflection groups *Trans. Am. Math. Soc.* **311** 167–83

[8] Ha Z N C 1994 Exact dynamical correlation functions of Calogero–Sutherland model and one-dimensional fractional statistics *Phys. Rev. Lett.* **73** 1574–7

[9] Hikami K and Wadati M 1997 Topics in quantum integrable system *Important Development in Soliton Theory* ed A S Fokas and V E Zakharov (Berlin: Springer)

[10] Humphreys J 1990 *Reflection Groups and Coxeter Groups* (Cambridge: Cambridge University Press)

[11] Kato Y 1998 Hole dynamics of the one-dimensional supersymmetric t-J model with a long-range interaction: an exact result *Phys. Rev. Lett.* **14** 5402–5

[12] Komori Y and Hikami K 1998 Affine R-matrix and the generalized elliptic Ruijsenaars models *Lett. Math. Phys.* **43** 335–46

[13] Komori Y 1998 Notes on the elliptic Ruijsenaars operators *Lett. Math. Phys.* **46** 147–55

[14] Komori Y 1999 Theta functions associated with the affine root systems and the elliptic Ruijsenaars operators *Preprint* math QA/9910003

[15] Lapointe L and Vinet L 1995 A Rodrigues formula for the Jack polynomials and the Macdonald–Stanley conjecture *Int. Math. Res. Not.* **9** 419–24

[16] Lapointe L and Vinet L 1997 Rodrigues formulae for the Macdonald polynomials *Adv. Math.* **130** 261–79

[17] Macdonald I G 1972 The Poincaré series of a Coxeter group *Math. Ann.* **199** 161–74

[18] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)

[19] Macdonald I G 1995 Affine Hecke algebras and orthogonal polynomials *Séminaire Bourbaki* **47** 189–297

[20] Nishino A, Ujino H and Wadati M 1999 Rodrigues formula for the nonsymmetric multivariable Laguerre polynomial *J. Phys. Soc. Japan* **68** 797–802

[21] Nishino A, Ujino H and Wadati M 1999 An algebraic approach for the non-symmetric Macdonald polynomial *Nucl. Phys.* B **558** 589–603

[22] Nishino A, Ujino H, Komori Y and Wadati M 2000 Rodrigues formulae for the non-symmetric multivariable polynomials associated with the BC_N-type root system *Nucl. Phys.* B **571** 632–48

[23] Nishino A, Komori Y, Ujino H and Wadati M Symmetrization of the nonsymmetric Macdonald polynomials and Macdonald's inner product identities *Preprint*

[24] Olshanetsky M A and Perelomov A M 1983 Quantum integrable systems related to Lie algebras *Phys. Rep.* **94** 313–404

[25] Opdam E M 1989 Some applications of hypergeometric shift operators *Invent. Math.* **98** 1–18

[26] Opdam E M 1998 Lectures on Dunkl operators *Preprint* math RT/9812007

[27] Sahi S 1999 Nonsymmetric Koorwinder polynomials and duality *Ann. Math.* **150** 267–82

[28] Sahi S 1998 A new formula for weight multiplicities and characters *Preprint* math QA/9802127

[29] Sutherland B 1971 Exact results for a quantum many-body problem in one dimension I *Phys. Rev.* A **4** 2019–21

[30] Sutherland B 1972 Exact results for a quantum many-body problem in one dimension II *Phys. Rev.* A **5** 1372–6

[31] Uglov D 1998 Yangian Gelfand–Zetlin bases, $\mathfrak{gl}_N$-Jack polynomials and computation of dynamical correlation functions in the spin Calogero–Sutherland model *Commun. Math. Phys.* **191** 663–96

[32] Ujino H and Wadati M 1997 Orthogonality of the Hi-Jack polynomials associated with the Calogero model *J. Phys. Soc. Japan* **66** 345–50

[33] Ujino H and Wadati M 1999 Rodrigues formula for the nonsymmetric multivariable Hermite polynomial *J. Phys. Soc. Japan* **68** 391–5

[34] Ujino H and Nishino A 1999 Rodrigues formulae for nonsymmetric multivariable polynomials associated with quantum integrable systems of Calogero–Sutherland type *Proc. Int. Workshop on Special Functions—Asymptotics, Harmonic Analysis and Mathematical Physics (City University, Hong Kong, June 1999)*